The following theorem (see Sundaram [495] and Topkis [516] for proofs) shows that the property of decreasing differences and supermodularity can be used to establish such parametric monotonicity.

THEOREM C.1 *Suppose that (i) the optimization problem (C.5) has at least one optimal solution for each* $\boldsymbol{\theta} \in \Theta$, *(ii)* $\boldsymbol{f}$ *satisfies decreasing differences in* $(\mathbf{x}, \boldsymbol{\theta})$, *and (iii)* $\boldsymbol{f}$ *is supermodular in* x *for each* $\boldsymbol{\theta} \in \Theta$. *Then for each* $\boldsymbol{\theta}$ *there exist a greatest optimal solution* $\mathbf{x}^*(\boldsymbol{\theta}) \in D^*(\boldsymbol{\theta})$. *This greatest optimal solution is nondecreasing in the parameters* $\boldsymbol{\theta}$; *that is,* $\mathbf{x}^*(\boldsymbol{\theta}_1) \geq^* (\boldsymbol{\theta}_2)$ *for all* $\boldsymbol{\theta}_1 > \boldsymbol{\theta}_2$.

This result says that higher values of $\boldsymbol{\theta}$ lead to higher optimal decisions $\mathbf{x}^*$. Corresponding definitions of decreasing difference and submodularity are used to show when optimal solutions are decreasing in a given parameter.

Linear Programs

An optimization problem is called a *linear program* if the objective function and all the equality and inequality constraints are defined by linear functions. That is, $f(\mathbf{x}) = \mathbf{c}^{\mathsf{T}}\mathbf{x}$ for some vector $\mathbf{c} \in \Re^n$, and the constraint set is of the form $X = \{\mathbf{Ax} = \mathbf{b}\}$ or $X = \{\mathbf{Ax} \leq \mathbf{b}\}$ (or combinations of inequality and equality constraints). Since we can always write an equality constraint $\mathbf{a}^{\mathsf{T}}\mathbf{x} = \mathbf{b}$ as two inequality constraints, $\mathbf{a}^{\mathsf{T}}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{a}^{\mathsf{T}}\mathbf{x} \geq \mathbf{b}$, and we can always write a variable $\mathbf{x}$ as $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$, where $\mathbf{x}^+ \geq \mathbf{0}$ and $\mathbf{x}^- \geq \mathbf{0}$ - any linear program can be converted into the form

$$\begin{aligned} \max \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq 0. \end{aligned} \tag{C.6}$$

Many problems can be expressed as linear programs, and there are specialized, highly efficient algorithms for solving them; hence, they warrant special attention. We have the following proposition, which shows that we can restrict our search for optimal solutions to extreme point solutions:

PROPOSITION C.12 *If the linear program (C.6) has an optimal solution, then it has an optimal solution that is an extreme point of the set* $X = \{\mathbf{x} : \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$.

The popular *simplex algorithm* for solving linear programs is based on searching the extreme points of the set *X*.

The linear program (C.6) has an associated *dual* linear program (or dual problem) defined by

$$\begin{aligned} \min \quad & \boldsymbol{\pi}^{\mathsf{T}}\mathbf{b} \\ \text{s.t.} \quad & \boldsymbol{\pi}^{\mathsf{T}}\mathbf{A} \geq \mathbf{c}^{\mathsf{T}} \\ & \boldsymbol{\pi} \geq 0. \end{aligned} \tag{C.7}$$

The original problem (C.6) is called the *primal* linear program (or primal problem). The primal and dual problems are related as follows:

PROPOSITION C.13 *If either the primal problem (C.6) or the dual problem (C.7) has a finite optimal solution, then so does the other, and the optimal objective function values are equal. Moreover, if the primal is unbounded, then the dual is infeasible, and if the dual is unbounded, then the primal is infeasible.*